Labeled Plane Trees and Increasing Plane Trees

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Abstract: This note is dedicated to presenting a polynomial analogue of $(n+1)!C_n = 2^n(2n-1)!!$ (with C_n as the n-th Catalan number) in the context of labeled plane trees and increasing plane trees, based on the definition of improper edges in labeled plane trees. A new involution on labeled plane trees is constructed to establish this identity, implying that the number of improper edges and the number of proper edges are equidsitributed over the set of labeled plane trees.

Keywords: labeled plane trees, increasing plane trees, improper edges, bijections, Stirling permutations

AMS Classification: 05A15, 05A19, 05C30

1 Introduction

The main objective of this paper is to build a connection between labeled plane trees and increasing plane trees based on the notation of improper edges in labeled plane trees. Recall that a plane tree is a rooted tree in which the children of each node are linearly ordered. It is well-known that the number of labeled plane trees with n edges is $(n+1)!C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number. An increasing plane tree, often referred to as a plane recursive tree, is a tree in which the node labels increase along any path from the root, and the children of every node are linearly ordered. Note that plane recursive trees also appear in literature under the names plane-oriented recursive trees, heap-ordered trees, and sometimes also as scale-free trees. It is not hard to see that the number of increasing plane trees with n edges is $(2n-1)!! = 1\cdot 3\cdots(2n-1)$. It is clear that

$$(n+1)!C_n = 2^n(2n-1)!!. \tag{1.1}$$

The main objective of this paper is to give a polynomial analogue of (1.1) based on the notion of improper edges in labeled plane trees. Improper edges of labeled plane trees were introduced by Guo and Zeng [6] to offer a combinatorial interpretation of the generalized Ramanujan polynomials defined by Chapoton [1]. It is worth noting that Shor [10] and Dumont and Ramamonjisoa [4] independently defined the improper edge of labeled rooted trees, which resulted in a refinement of Cayley's formula. Zeng [12] later discovered a link between Shor's polynomials and the Ramanujan polynomials, see Chen, Fu and Wang [2] and Chen and Yang [3] for more details. Subsequently, Guo and Zeng [6] extended the notation of improper edges from labeled rooted trees to labeled plane trees.

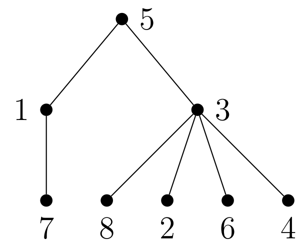

Figure 1: A labeled plane tree with 7 edges.

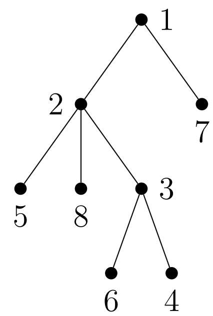

Figure 2: An increasing plane tree with 7 edges.

Let $\mathcal{P}_n$ (resp. $\mathcal{O}_n$) denote the set of labeled plane trees (resp. labeled plane trees with root 1) with n edges. Given a vertex j of a labeled plane tree T, let $\beta(j)$ be the smallest label on the subtree with root j. Suppose that the vertex j in T has k children: $j_1, j_2, \ldots, j_k$ (ordered left to right). We call the edge (j, j_i) is imporper if

$$\beta(j_i) < \min\{j, \beta(j_{i+1}), \ldots, \beta(j_k)\}, \tag{1.2}$$

otherwise we call it a proper edge.

For example, the labeled plane tree $T \in \mathcal{P}_7$ depicted in Fig. 1 has three improper edges: $(5,1), (5,3), (3,2)$. All other edges in T are proper.

Let $\mathrm{impr}(T)$ be the number of improper edges of T and let $\mathrm{prop}(T)$ be the the number of proper edges of T. For the labeled plane tree $T \in \mathcal{P}_7$ shown in Fig. 1, this gives $\mathrm{impr}(T) = 3$ and $\mathrm{prop}(T) = 4$.

It is worth noting that every edge in an increasing plane tree is proper. Conversely, any labeled plane tree without improper edges must be an increasing plane tree.

For example, in the increasing plane tree $T \in \mathcal{P}_7$ shown in Fig. 2, we have $\mathrm{impr}(T) = 0$ and $\mathrm{prop}(T) = 7$.

In this paper, we consider the polynomials that incorporate both improper edges and proper edges of labeled plane trees:

$$P_n(x,y) = \sum_{T \in \mathcal{P}_n} x^{\mathrm{impr}(T)} y^{\mathrm{prop}(T)},$$

$$O_n(x,y,t) = \sum_{T \in \mathcal{O}_n} x^{\mathrm{impr}(T)} y^{\mathrm{prop}(T) - \deg_T(1)} t^{\deg_T(1)},$$

where $\deg_T(1)$ denotes the number of children of root 1.

For example,

$$P_1(x,y) = x + y;$$

$$P_2(x,y) = 3x^2 + 6xy + 3y^2;$$

$$P_3(x,y) = 15x^3 + 45x^2y + 45xy^2 + 15y^3.$$

and

$$O_1(x,y,t) = ty;$$

$$O_2(x,y,t) = 2t^2y^2 + tx + ty;$$

$$O_3(x,y,t) = 6t^3 + 6t^2x + 6t^2y + 3tx^2 + 6txy + 3ty^2.$$

By constructing an involution on labeled plane trees, we show the following consequence:

Theorem 1.1. *For $n \geq 1$,*

$$P_n(x,y) = (2n-1)!!(x+y)^n, \tag{1.3}$$

$$O_n(x,y,t) = \sum_{r=1}^{n} t^r S_{n,r} (x+y)^{n-r}, \tag{1.4}$$

where $S_{n,r}$ counts the number of increasing plane trees with n edges so that the degree of 1 is r.

It should be noted that $S_{n,r}$ appeared in Theorem 1.1 also counts the number of Stirling permutations on $\{1,1,2,2,\ldots,n,n\}$ with r blocks. For $n \geq 1$, let $[n]_2$ denote the multiset $\{1,1,2,2,\ldots,n,n\}$. Recall that a Stirling permutation on $[n]_2$ introduced by Gessel and Stanley [5] is defined as a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n}$ on $[n]_2$ such that, for any i, the elements between the two occurrences of i in σ, if any, are greater than i. For example, $\sigma = 6\,6\,3\,4\,5\,5\,4\,3\,1\,1\,2\,7\,7\,2$ is a Stirling permutation on $[7]_2$. A block of a Stirling permutation $\sigma = \sigma_1 \cdots \sigma_{2n}$ on $[n]_2$ is a substring $a_i \ \cdots \ a_j$ with $a_i = a_j$ that is not contained in any larger such substring. For example, the Stirling permutation $\sigma = 6\,6\,3\,4\,5\,5\,4\,3\,1\,1\,2\,7\,7\,2$ has four blocks as seen in its block decomposition: $[6\,6][3\,4\,5\,5\,4\,3][1\,1][2\,7\,7\,2]$. For more details, please see Janson, Kuba and Panholzer [8] and Remmel and Wilson [9].

Janson [7] showed that there exists a bijection between the set of increasing plane trees with n edges and r children of root 1 and the set of Stirling permutations on $[n]_2$ with r blocks. The desired bijection is the well-known *depth first walk* of a rooted plane trees. Recall that the depth first walk of a rooted plane tree started at the root, goes first to the leftmost child of the root, explores that branch (recursively, using the same rules), returns to the root, and continues with the next child of the root, until there are no more children left.

For the increasing plane tree depicted in Fig. 2, applying the depth-first walk yields the corresponding Stirling permutation:

$$1\,4\,4\,7\,7\,2\,5\,5\,3\,3\,2\,1\,6\,6,$$

which has two blocks as shown in its block decomposition:

$$[1\,4\,4\,7\,7\,2\,5\,5\,3\,3\,2\,1][6\,6].$$

Let

$$S_n(t)=\sum_{r=1}^{n}S_{n,r}t^r, \tag{1.5}$$

where $S_{n,r}$ counts the number of increasing plane trees with n edges so that the degree of 1 is r or counts the number of Stirling permutations on $\{1,1,2,2,\ldots,n,n\}$ with r blocks. Evidently, $S_n(1)=(2n-1)!!$, so we have

$$\sum_{n=0}^{\infty}S_n(1)\frac{q^n}{n!}=\frac{1}{\sqrt{1-2q}}. \tag{1.6}$$

Using the compositional formula [11, Theorem 5.1.4], one can readily show that

$$\sum_{n=0}^{\infty}S_n(t)\frac{q^n}{n!}=\frac{1}{1-t+t\sqrt{1-2q}}. \tag{1.7}$$

To our knowledge, this result is new, as we have not encountered it in the literature.

Applying Theorem 1.1 to (1.6) and (1.7) yields that

Theorem 1.2. *We have*

$$\sum_{n=0}^{\infty}P_n(x,y)\frac{q^n}{n!}=\frac{1}{\sqrt{1-2(x+y)q}}, \tag{1.8}$$

$$\sum_{n=0}^{\infty}O_n(x,y,t)\frac{q^n}{n!}=\frac{x+y}{x+y-t+t\sqrt{1-2(x+y)q}}. \tag{1.9}$$

2 An involution on labeled plane trees

For a nonempty labeled plane tree $T \in \mathcal{P}_n$, let $e = (i, j)$ be an edge of T. Assume that i has p children, that is, $k_1, \ldots, k_{t-1}, j, k_{t+1}, \ldots, k_p$, where j is its t-th child from left to right. Let τ_i denote the subtree rooted by i. The plane tree T is divided into three parts according to the edge $e = (i, j)$ (see Fig. 3):

- $A(e)$: the forest formed by subtrees $\tau_{k_1}, \ldots, \tau_{k_{t-1}}$;
- $B(e)$: the forest formed by subtrees $\tau_{l_1}, \ldots, \tau_{l_q}$;
- $C(e)$: the forest formed by subtrees $\tau_{k_{t+1}}, \ldots, \tau_{k_p}$.

Note that the forests $A(e)$, $B(e)$ and $C(e)$ can be empty. Let $\hat{T}$ be the labeled plane tree obtained from T by swapping $\{i, C(e)\}$ and $\{j, B(e)\}$ (see Fig. 4) and set $\phi_e(T) = \hat{T}$.

It is clear that the map ϕ_e is an involution on $\mathcal{P}_n$. Moreover, the sets of edges in T and $\hat{T}$ are identical. The only differences between the two trees lie in the order of the edges and the vertex connections of the edge e.

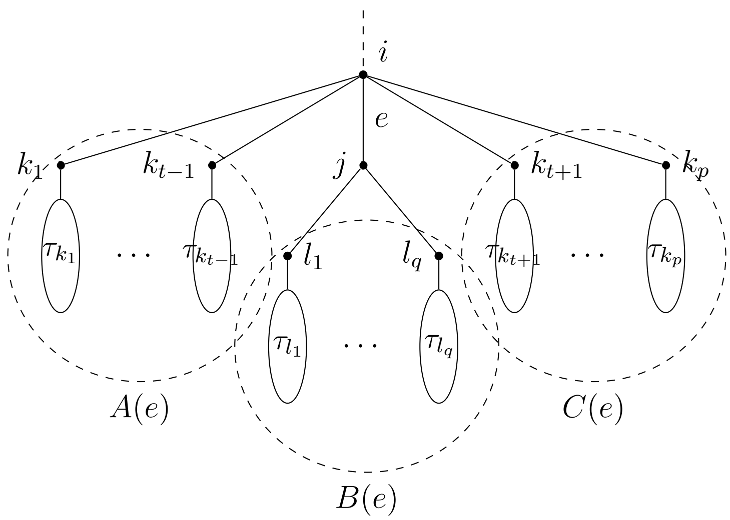

Figure 3: The decomposition of a labeled plane tree T based on edge e

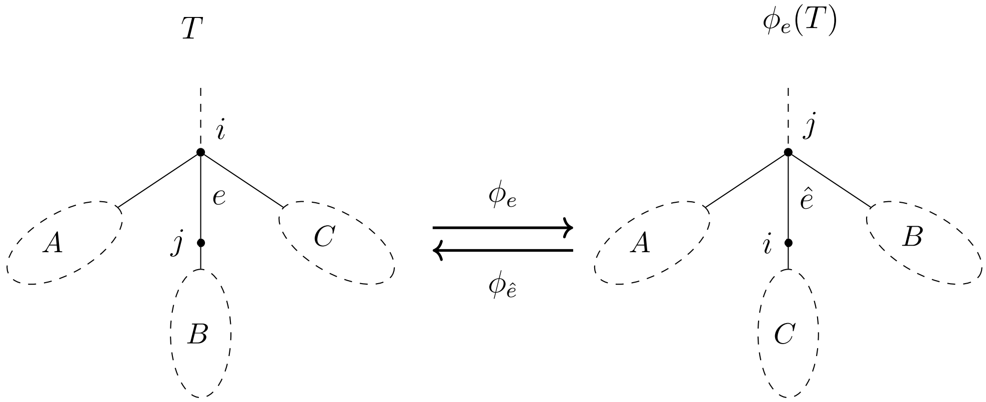

Figure 4: A visual illustration of the involution ϕ

The main property of this involution ϕ is contained in the following proposition:

Proposition 2.1. *For a labeled plane tree T and an edge e in T, define $\hat{T} := \phi_e(T)$. The proper or improper status of the edge e is reversed between T and $\hat{T}$, while all edges other than e preserve their proper or improper property.*

Proof. Let $a = (a_1, a_2)$ be any edge of T. As illustrated in Fig. 3, we decomposed the plane tree T into three parts based on the edge $a = (a_1, a_2)$. Let $A_T(a), B_T(a), C_T(a)$ be the forests formed by subtrees of T, as defined in the construction of the involution ϕ. Define $\mathcal{B}_T(a) = \{a_2, B_T(a)\}$ and $\mathcal{C}_T(a) = \{a_1, C_T(a)\}$. By definition, if a is a proper edge of T, then $\min \mathcal{B}_T(a) > \min \mathcal{C}_T(a)$, otherwise, $\min \mathcal{B}_T(a) < \min \mathcal{C}_T(a)$.

Similarly, for $\hat{T}$, we define $\mathcal{B}_{\hat{T}}(a)$ and $\mathcal{C}_{\hat{T}}(a)$ with respect to the edge a in $\hat{T}$. The proper/improper status of a in $\hat{T}$ is also determined by comparing the minimums of $\mathcal{B}_{\hat{T}}(a)$ and $\mathcal{C}_{\hat{T}}(a)$.

From the construction of the involution ϕ_e, we see

if $a = e$, then $\mathcal{B}_T(e) = \mathcal{C}_{\hat{T}}(e)$ and $\mathcal{C}_T(e) = \mathcal{B}_{\hat{T}}(e)$. This implies the proper/improper status of e is reversed between T and $\hat{T}$.

if $a \neq e$, then $\mathcal{B}_T(a) = \mathcal{B}_{\hat{T}}(a)$ and $\mathcal{C}_T(a) = \mathcal{C}_{\hat{T}}(a)$ Thus, the proper/improper status of the edge a remains unchanged in T and $\hat{T}$. This completes the proof. ∎

From the construction of ϕ, it is not hard to check that

Proposition 2.2. *Given a labeled plane tree T, the involutions ϕ commute for any pair of edges e_1, e_2 in T:*

$$\phi_{e_2}\phi_{e_1}(T) = \phi_{e_1}\phi_{e_2}(T).$$

3 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1 based on the involution ϕ defined in Section 2.

Proof of Theorem 1.1. Let $\mathcal{I}_n$ denote the set of increasing plane trees with n edges. To prove (1.3), we first define edge labelings for plane trees in $\mathcal{P}_n$ and $\mathcal{I}_n$.

Let T be a labeled plane tree with n edges. We assign a label to each edge of T: improper edges are labeled by x, and proper edges are labeled by y. The weight of T, denoted by $\mathrm{wt}(T)$ is defined to be the product of these edge labels. It is clear that,

$$P_n(x, y) = \sum_{T \in \mathcal{P}_n} \mathrm{wt}(T).$$

For the plane tree $T \in \mathcal{I}_n$, we introduce a free labeling: each edge may be labeled either x or y. The weight of $T \in \mathcal{I}_n$, denoted by $\widetilde{\mathrm{wt}}(T)$, is the product of its edge labels. It follows that

$$\sum_{T \in \mathcal{I}_n} \widetilde{\mathrm{wt}}(T) = (2n-1)!!(x+y)^n.$$

Proving (1.3) is thus equivalent to showing

$$\sum_{T \in \mathcal{P}_n} \mathrm{wt}(T) = \sum_{T \in \mathcal{I}_n} \widetilde{\mathrm{wt}}(T). \tag{3.1}$$

Let T be a labeled plane tree in $\mathcal{P}_n$ with k improper edges. These improper edges are denoted by $e_1, e_2, \ldots, e_k$ in the order they appear in the depth-first walk of the rooted plane tree. Applying the involutions $\phi_{e_1}, \ldots, \phi_{e_k}$ to T successively (preserving edge labels), we get the labeled plane tree $\hat{T} = \phi_{e_k}\phi_{e_{k-1}} \cdots \phi_{e_1}(T)$. Define $\Phi = \phi_{e_k}\phi_{e_{k-1}} \cdots \phi_{e_1}$, so that $\hat{T} = \Phi(T)$, see Fig. 5. By Proposition 2.1, $\hat{T}$ contains no improper edges, and so $\hat{T} \in \mathcal{I}_n$ (the set of increasing plane trees with n edges). Moreover, $\mathrm{wt}(T) = \widetilde{\mathrm{wt}}(\hat{T})$, where the x-labeled edges in $\widetilde{\mathrm{wt}}(\hat{T})$ correspond to the improper edges of T.

Conversely, let $\hat{T} \in \mathcal{I}_n$ with the weight $\widetilde{\mathrm{wt}}(\hat{T})$. Assume that there are k edges labeled by x. Let $\hat{e}_1, \hat{e}_2, \ldots \hat{e}_k$ be its x-labeled edges in the depth-first walk order. By successively applying the involutions $\phi_{\hat{e}_1}, \ldots, \phi_{\hat{e}_k}$ to $\hat{T}$ (preserving labels), we obtain the labeled plane tree $T = \phi_{\hat{e}_k}\phi_{\hat{e}_{k-1}} \cdots \phi_{\hat{e}_1}(\hat{T})$. Define $\Psi = \phi_{\hat{e}_k}\phi_{\hat{e}_{k-1}} \cdots \phi_{\hat{e}_1}$, which gives $T = \Psi(\hat{T})$. By Proposition 2.1, T has exactly k improper edges (all labeled x), and so $\mathrm{wt}(T) = \widetilde{\mathrm{wt}}(\hat{T})$. Furthermore, by Proposition 2.2, we derive that $T = \Psi(\Phi(T))$ for all $T \in \mathcal{P}_n$ and $\hat{T} = \Phi(\Psi(\hat{T}))$ for all $\hat{T} \in \mathcal{I}_n$. This establishes a weight-preserving bijection between the two sums in (3.1), completing the proof of (3.1) and thus of (1.3).

The proof of (1.4) is similar to that of (1.3). It suffices to modify the edge labelings for plane trees in $\mathcal{O}_n$ and $\mathcal{I}_n$ as follows: For any $T \in \mathcal{O}_n$, label the edges connected to vertex 1 with t and label all other edges with x or y using the same rule as in $\mathcal{P}_n$ (i.e., x for improper edges, y for proper edges). The weight of $T \in \mathcal{O}_n$, denoted by $\mathrm{wt}^r(T)$, is defined to be the product of these edge labels. Clearly,

$$O_n(x,y,t) = \sum_{T\in\mathcal{O}_n} \mathrm{wt}^r(T).$$

For increasing plane trees $T \in \mathcal{I}_n$, we introduce a modified free labeling: Edges connected to vertex 1 are labeled by t, while all other edges may be labeled either x or y. The weight of $T \in \mathcal{I}_n$, denoted by $\widetilde{\mathrm{wt}}^r(T)$, is the product of its edge labels. It follows that

$$\sum_{r=1}^{n} t^r S_{n,r}(x+y)^{n-r} = \sum_{T\in\mathcal{I}_n} \widetilde{\mathrm{wt}}^r(T).$$

Thus, establishing (1.4) reduces to showing that

$$\sum_{T\in\mathcal{O}_n} \mathrm{wt}^r(T) = \sum_{T\in\mathcal{I}_n} \widetilde{\mathrm{wt}}^r(T). \tag{3.2}$$

The bijection Φ used in the proof of (3.1) can be readily adapted to establish (3.2), and consequently (1.4). This completes the proof of Theorem 1.1. ∎

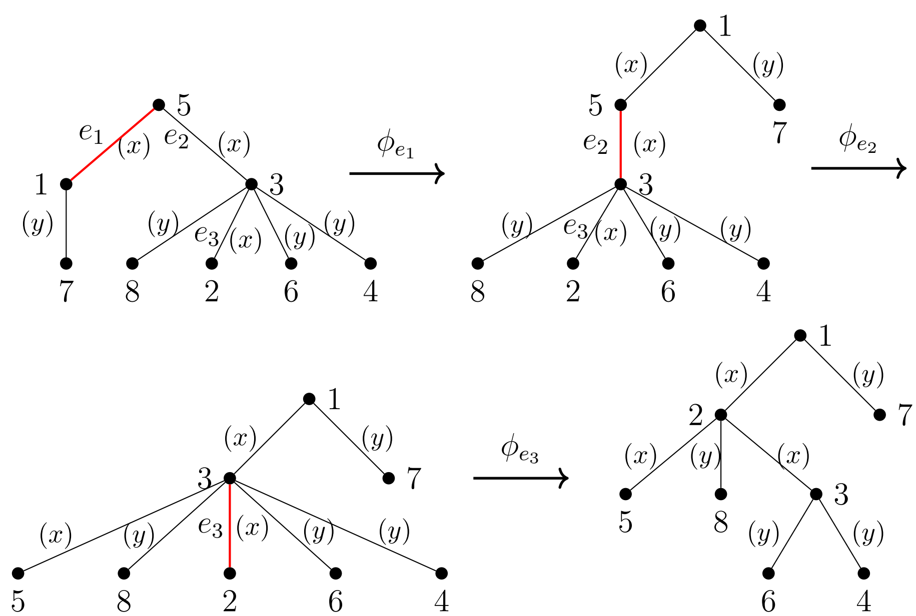

Figure 5: An example of the bijection $\Phi = \phi_{e_1}\phi_{e_2}\phi_{e_3}$

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